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A Modified Analysis of Magnetohydrodynamic Thermal/Thermohaline Convection

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The problem of modified magnetohydrodynamic thermohaline convection of G. Veronis' type (*J. Mar. Res.* **23** (1965), 1–17) is exactly solved on the lines shown by M. B. Banerjee *et al.* (*J. Math. Anal. Appl.* **144** (1989), 356–366) for the magnetohydrodynamic thermal convection. Only the overstable case is treated and the expressions for the critical Rayleigh number and the frequency of oscillations are derived. Further, sufficient conditions for the validity of overstability and the principle of exchange of stabilities (PES), respectively, are obtained and the region for arresting the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, is given. Results for the corresponding modified problems of Bénard convection, magnetohydrodynamic Bénard convection, generalized Bénard convection, magnetohydrodynamic generalized Bénard convection, and thermohaline convection follow as a consequence. © 1991 Academic Press, Inc.

1. INTRODUCTION

The problem of thermohaline convection, aside from its many applications in the fields of oceanography, astrophysics, chemical engineering, etc., has recently received considerable attention due to its interesting complexities as a double-diffusive phenomenon. Banerjee *et al.* [3] presented a modified analysis of thermal/thermohaline instability of a liquid layer heated underside (Bénard convection). They ensured that the linear theoretical explanation of the phenomenon of gravity-dominated thermal instability in a liquid layer heated underside depends not only upon the Rayleigh number, which is proportional to the uniform temperature difference maintained across the layer, but also upon another parameter that takes care of the fact that a relatively hotter layer with its heat diffusivity apparently increased/decreased as a consequence of an actual decrease/increase (depending upon the fluid) in its specific heat at constant volume must exhibit Bénard convection at a higher/lower temperature difference across the layer and hence at a higher/lower Rayleigh number than a

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cooler layer under almost identical conditions otherwise. Further, this qualitative effect is not quantitatively insignificant. Detailed consequences are worked out for the onset of thermohaline instability in a liquid layer heated underside. In a subsequent paper Banerjee *et al.* [4] carried out similar investigations in the framework of rotation. Katoch [5] examined the problem in the presence of a uniform vertical magnetic field. However, as in Banerjee *et al.* [6], it can be shown that Katoch's solution for the vertical velocity in the overstable case like that of Chandrasekhar's solution for the simple magnetohydrodynamic Bénard problem with free and perfectly conducting boundaries is not correct and as a consequence the results derived by him cannot be relied upon. The present paper presents a modified analysis of the problem of magnetohydrodynamic thermohaline convection of Veronis' [1] type and derives a correct solution of the problem in the overstable case which takes care of the fact that a valid solution for the vertical veloctiy must be such that its fourth order derivative does not vanish on one of the boundaries at least. Further, sufficient conditions for the validity of overstability and the PES, respectively, are obtained and the region for arresting the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, is given. Results for the corresponding modified problems of Bénard convection, magnetohydrodynamic Bénard convection, generalized Bénard convection, magnetohydrodynamic generalized Bénard convection, and thermohaline convection follow as a consequence.

2. Modified Simplified Equations for Magnetohydrodynamic Thermal/Thermohaline Convection

Following Banerjee *et al.* [3], the modified simplified equations governing Bénard convection ($S=0=\Delta\rho'$), generalized Benard convection ($\tau_0=0$), and thermohaline convection of Veronis' type under the effect of a uniform vertical magnetic field are given by

$$\frac{\partial U_j}{\partial x_j}=0, \tag{1}$$

$$\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}-\frac{\mu_e}{4\pi\rho_0}H_j\frac{\partial H_i}{\partial x_j}$$
$$=-\frac{\partial}{\partial x_i}\left(\frac{p}{\rho_0}+\frac{\mu_e|H|^2}{8\pi\rho_0}\right)+\left(1+\frac{\Delta\rho}{\rho_0}+\frac{\Delta\rho'}{\rho_0}\right)X_i+\nu_0\nabla^2U_i, \tag{2}$$

$$(1-\alpha_2 T)\left(\frac{\partial T}{\partial t}+U_j\frac{\partial T}{\partial x_j}\right)+\hat{\alpha}_2 T\left(\frac{\partial S}{\partial t}+U_j\frac{\partial S}{\partial x_j}\right)=\kappa_0\nabla^2 T, \tag{3}$$

$$\frac{\partial S}{\partial t}+U_j\frac{\partial S}{\partial x_j}=\tau_0\nabla^2 S, \tag{4}$$

$$\frac{\partial H_\iota}{\partial t}+U_j\frac{\partial H_\iota}{\partial x_j}=H_j\frac{\partial U_\iota}{\partial x_j}+\eta_0\nabla^2 H_\iota, \tag{5}$$

$$\frac{\partial H_j}{\partial x_j}=0, \tag{6}$$

and

$$\rho=\rho_0[1+\alpha(T_0-T)-\hat{\alpha}(S_0-S)], \tag{7}$$

where $\iota=1, 2, 3$, x_ι's, u_ι's, H_ι's, and X_ι's are respectively the cartesian coordinates x, y, z, velocity components u, v, w, magnetic field components H_1, H_2, H_3, and external force components 0, 0, $-g$; t is the time, p is the pressure; $\Delta\rho$ and $\Delta\rho'$ are respectively the variations in the density due to temperature and concentration; ρ_0, v_0, κ_0, τ_0, and η_0 are respectively the values of the density, viscosity, thermal diffusivity, mass diffusivity, and magnetic diffusivity at the lower boundary; α_2 and $\hat{\alpha}_2$ are respectively the coefficients of specific heat variation due to temperature and concentration variations; α and α' are respectively the coefficients of volume expansion due to temperature and concentration; T is the temperature; and S is the concentration.

3. Modified Analysis of Magnetohydrodynamic Bénard Convection/Generalized Bénard Convection and Thermohaline Convection

Following the usual steps of linear stability theory the system of equations (1)–(7) yields the following nondimensional perturbation equations:

$$(D^2-a^2)\left(D^2-a^2-\frac{p}{\sigma}\right)w=R_1a^2\theta-\frac{R_2}{R_3}a^2\phi-Q\,D(D^2-a^2)h_z, \tag{8}$$

$$[D^2-a^2-p(1-\alpha_2T_0)]\theta-T_0\hat{\alpha}_2p\phi=-(1-\alpha_2T_0)w-T_0\hat{\alpha}_2R_3w, \tag{9}$$

$$[\tau(D^2-a^2)-p]\phi=-R_3w, \tag{10}$$

and

$$\left(D^2-a^2-\frac{p\sigma_1}{\sigma}\right)h_z=-Dw. \tag{11}$$

Solutions of Eqs. (8)–(11) must be sought subject to the following boundary conditions:

$$w=0=D^2w=D\theta=D\phi=h_z \quad \text{at} \quad z=-\tfrac{1}{2} \text{ and } z=+\tfrac{1}{2}, \tag{12}$$

or

$$w=0=Dw=D\theta=D\phi=h_z \quad \text{at} \quad z=-\tfrac{1}{2} \text{ and } z=+\tfrac{1}{2}, \tag{13}$$

or

$$w=0=D^2w=D\theta=D\phi=Dh_z \pm ah_z \quad \text{at} \quad z=-\tfrac{1}{2} \text{ and } z=+\tfrac{1}{2}, \tag{14}$$

or

$$w=0=Dw=D\theta=D\phi=Dh_z \pm ah_z \quad \text{at} \quad z=-\tfrac{1}{2} \text{ and } z=+\tfrac{1}{2}, \tag{15}$$

or

$$w=0=D^2w=\theta=\phi=h_z \quad \text{at} \quad z=-\tfrac{1}{2} \text{ and } z=+\tfrac{1}{2}, \tag{16}$$

or

$$w=0=Dw=\theta=\phi=h_z \quad \text{at} \quad z=-\tfrac{1}{2} \text{ and } z=+\tfrac{1}{2}, \tag{17}$$

or

$$w=0=D^2w=\theta=\phi=Dh_z \pm ah_z \quad \text{at} \quad z=-\tfrac{1}{2} \text{ and } z=+\tfrac{1}{2}, \tag{18}$$

or

$$w=0=Dw=\theta=\phi=Dh_z \pm ah_z \quad \text{at} \quad z=-\tfrac{1}{2} \text{ and } z=+\tfrac{1}{2}. \tag{19}$$

In the above equations z is the vertical coordinate, $z=-\frac{1}{2}$ and $z=+\frac{1}{2}$ represent the two boundaries, $D=d/dz$, w is the vertical velocity, θ is the temperature, ϕ is the concentration, h_z is the vertical magnetic field, a^2 is the square of the wave number, σ is the thermal Prandtl number, R_1 (>0) is the Rayleigh number, R_2 (>0) is the concentration Rayleigh number, R_3 is the ratio of concentration gradient to temperature gradient, Q is the Chandrasekhar number, $p=p_r+\iota p_\iota$ is the complex growth rate, τ is the ratio of mass diffusivity to heat diffusivity, and σ_1 is the magnetic Prandtl number.

It is important to note here that the system of equations and boundary conditions as given by Eqs. (8)–(19) correspond to

(i) simple Bénard convection if $\alpha_2=0=\hat{\alpha}_2=R_2=Q$,

(ii) modified simple Bénard convection if $\hat{\alpha}_2=0=R_2=Q$,

(iii) modified simple magnetohydrodynamic Bénard convection if $\hat{\alpha}_2=0=R_2$,

(iv) generalized Bénard convection if $\hat{\alpha}_2=0=\alpha_2=\tau=Q$,

(v) modified generalized Bénard convection if $\hat{\alpha}_2 = 0 = \tau = Q$,

(vi) modified hydromagnetic generalized Bénard convection if $\hat{\alpha}_2 = 0 = \tau$,

(vii) thermohaline convection of Veronis' type if $\alpha_2 = 0 = \hat{\alpha}_2 = Q$,

(viii) modified thermohaline convection of Veronis' type if $Q = 0$.

4. Mathematical Analysis

Systems of equations (8)–(11) together with either of the boundary conditions (12)–(19) constitute an eigenvalue problem for p for given values of the other parameters and a given state of the system is stable, neutral, or unstable according to whether p_r, the real part of p, is negative, zero, or positive. Further, if $p_r = 0$ implies $p_i = 0$ for all wave numbers a^2, then the principle of exchange of stabilities (PES) is valid; otherwise we have overstability at least when instability sets in as certain modes.

(a) *Characterizations of the Marginal State*

THEOREM 1. *If* $(p, w, \theta, \phi, h_z)$ *is a solution of equations* (8)–(11) *together with either of the boundary conditions* (12)–(19) *and* $\tau = 0$, *then*

$$p \neq 0.$$

Proof. For $\tau = 0$, let $p = 0$ be allowed if possible. It then follows from Eq. (10) that

$$w \equiv 0. \tag{20}$$

Equations (9) and (11) now become

$$(D^2 - a^2)\theta = 0, \tag{21}$$

and

$$(D^2 - a^2)h_z = 0. \tag{22}$$

The only solutions of Eqs. (21) and (22) satisfying the relevant boundary conditions are

$$\theta \equiv 0, \tag{23}$$

and

$$h_z \equiv 0. \tag{24}$$

It now follows from Eq. (8) that

$$\phi \equiv 0. \tag{25}$$

Equations (20), (23)–(25) imply that p cannot be zero. This completes the proof of the theorem.

Theorem 1 implies that PES is not valid for the generalized magnetohydrodynamic Bénard convection when considered in the present generalized framework and this establishes the result due to Banerjee *et al.* [7] on a more firm basis.

THEOREM 2. *If* $(p, w, \theta, \phi, h_z)$ *is a solution of equations* (8)–(11) *together with either of the boundary conditions* (12)–(19) *and* $R_1 \leqslant Q\pi^2/(1+(\hat{\alpha}_2 R_3 - \alpha_2)T_0)$, *then*

$$p_r = 0 \Rightarrow p_i \neq 0.$$

Proof. If possible let $p_i = 0$ be allowed so that $p = 0$ and Eqs. (8)–(11) assume the forms

$$(D^2 - a^2)^2\, w = R_1 a^2 \theta - \frac{R_2}{R_3} a^2 \phi - Q\, D(D^2 - a^2) h_z, \tag{26}$$

$$(D^2 - a^2)\theta = -[1 + (\hat{\alpha}_2 R_3 - \alpha_2) T_0]\, w, \tag{27}$$

$$\tau(D^2 - a^2)\phi = -R_3 w, \tag{28}$$

and

$$(D^2 - a^2) h_z = -Dw. \tag{29}$$

Using Eq. (29), Eq. (26) can be written as

$$(D^2 - a^2)^2\, w = R_1 a^2 \theta - \frac{R_2}{R_3} a^2 \phi + Q\, D^2 w. \tag{30}$$

Multiplying Eqs. (30), (27), and (28) by w^*, $(R_1 a^2/(1 + (\hat{\alpha}_2 R_3 - \alpha_2) T_0))\theta^*$ and $-(R_2 a^2/R_3^2)\phi^*$ ($*$ indicates complex conjugation), respectively, adding the resulting equations, and integrating the equation so obtained over the vertical range of z by parts an appropriate number of times with the help of either of the boundary conditions (12)–(19), we get

$$\int_{-1/2}^{1/2} |(D^2 - a^2) w|^2\, dz + Q \int_{-1/2}^{1/2} |Dw|^2\, dz + \frac{\tau R_2 a^2}{R_3^2} \int_{-1/2}^{1/2} (|D\phi|^2 + a^2\, |\phi|^2)\, dz$$
$$= \frac{R_1 a^2}{1 + (\hat{\alpha}_2 R_3 - \alpha_2) T_0} \int_{-1/2}^{1/2} (|D\theta|^2 + a^2\, |\theta|^2)\, dz. \tag{31}$$

Multiplying Eq. (27) by its complex conjugate and integrating over the range of z, we get

$$\int_{-1/2}^{1/2} |D^2\theta|^2\, dz + a^4 \int_{-1/2}^{1/2} |\theta|^2\, dz + 2a^2 \int_{-1/2}^{1/2} |D\theta|^2\, dz$$
$$= [1 + (\hat{\alpha}_2 R_3 - \alpha_2) T_0]^2 \int_{-1/2}^{1/2} |w|^2\, dz. \tag{32}$$

Equation (32) implies that

$$a^2 \int_{-1/2}^{1/2} (|D\theta|^2 + a^2 |\theta|^2)\, dz < [1 + (\hat{\alpha}_2 R_3 - \alpha_2) T_0]^2 \int_{-1/2}^{1/2} |w|^2\, dz, \tag{33}$$

which upon using the Rayleigh–Ritz [8] inequality, namely,

$$\int_{-1/2}^{1/2} |w|^2\, dz \leqslant \frac{1}{\pi^2} \int_{-1/2}^{1/2} |Dw|^2\, dz \qquad \left(\text{since } w\left(-\frac{1}{2}\right) = 0 = w\left(\frac{1}{2}\right)\right), \tag{34}$$

gives

$$a^2 \int_{-1/2}^{1/2} (|D\theta|^2 + a^2 |\theta|^2)\, dz < \frac{[1 + (\hat{\alpha}_2 R_3 - \alpha_2) T_0]^2}{\pi^2} \int_{-1/2}^{1/2} |Dw|^2\, dz. \tag{35}$$

Equation (31) together with inequality (35) implies that

$$\int_{-1/2}^{1/2} |(D^2 - a^2) w|^2\, dz + \left[Q - \frac{R_1 \langle 1 + (\hat{\alpha}_2 R_3 - \alpha_2) T_0 \rangle}{\pi^2} \right] \int_{-1/2}^{1/2} |Dw|^2\, dz$$
$$+ \frac{\tau R_2 a^2}{R_3^2} \int_{-1/2}^{1/2} (|D\phi|^2 + a^2 |\phi|^2)\, dz < 0. \tag{36}$$

Inequality (36) obviously cannot hold under the condition of the theorem. Hence $p_i \neq 0$.

This completes the proof of the theorem.

Theorem 2 shows that for the problem under consideration, a sufficient condition for the validity of overstability is that $R_1 \leqslant Q\pi^2/(1 + (\hat{\alpha}_2 R_3 - \alpha_2) T_0)$ or equivalently a necessary condition for instability to set in as stationary convection is that $R_1 > Q\pi^2/(1 + (\hat{\alpha}_2 R_3 - \alpha_2) T_0)$. This result is valid for quite general boundary conditions. In particular, it follows that for the magnetohydrodynamic simple Bénard convection a necessary condition for instability to set in as stationary convection is that $R_1 > \pi^2 Q$ for all finite values of Q, a result predicted by Chandrasekhar [9]. However, the result that the critical Rayleigh number $R_{1c} \to \pi^2 Q$ as $Q \to \infty$, which constitutes

Chandrasekhar's famous $\pi^2 Q$-law and which was also predicted by him for quite general boundary conditions, cannot be derived from the present analysis and thus remains an open problem that is yet to be resolved.

THEOREM 3. *If* $(p, w, \theta, \phi, h_z)$ *is a solution of Eqs.* (8)–(11) *together with either of the boundary conditions* (16)–(19), $\hat{\alpha}_2 = 0$ *and* $R_2\sigma/2\tau^2\pi^4 + Q\sigma_1/\pi^2 \leqslant 1$, *then*

$$p_r \geqslant 0 \Rightarrow p_i = 0.$$

Proof. For $\hat{\alpha}_2 = 0$, Eq. (9) becomes

$$[D^2 - a^2 - p(1 - T_0\alpha_2)]\theta = -(1 - T_0\alpha_2)w. \tag{37}$$

Multiplying Eqs. (8), (10), (11), and (37) by w^*, $(R_2a^2/R_3)\phi^*$, $-Q(D^2 - a^2)h_z^*$, and $-(R_1a^2/(1 - T_0\alpha_2))\theta^*$, respectively, adding the resulting equations, integrating the equation so obtained over the vertical range of z by parts an appropriate number of times with the help of either of the boundary conditions (12)–(19), and finally equating the imaginary part of the resultant equation, we have for $p_i \neq 0$ the equation

$$\begin{aligned}\frac{1}{\sigma}\int_{-1/2}^{1/2} (|Dw|^2 + a^2\,|w|^2)\,dz + R_1a^2\int_{-1/2}^{1/2} |\theta|^2\,dz \\ = \frac{R_2a^2}{R_3^2}\int_{-1/2}^{1/2} |\phi|^2\,dz + \frac{Q\sigma_1}{\sigma}\left[\Gamma + \int_{-1/2}^{1/2} (|Dh_z|^2 + a^2\,|h_z|^2)\,dz\right],\end{aligned} \tag{38}$$

where

$$\Gamma = a[(|h_z|^2)_{-1/2} + (|h_z|^2)_{1/2}] \geqslant 0. \tag{39}$$

Equation (10) when multiplied by its complex conjugate and integrated over the vertical range of z yields

$$2a^2\int_{-1/2}^{1/2} |D\phi|^2\,dz < \frac{R_3^2}{\tau^2}\int_{-1/2}^{1/2} |w|^2\,dz, \tag{40}$$

which upon using inequality (34) and the analogous inequality involving ϕ (since $\phi(-\frac{1}{2}) = 0 = \phi(\frac{1}{2})$ presently) gives

$$a^2\int_{-1/2}^{1/2} |\phi|^2\,dz < \frac{R_3^2}{2\tau^2\pi^4}\int_{-1/2}^{1/2} |Dw|^2\,dz. \tag{41}$$

Further, as in Banerjee *et al.* [10], it follows from Eq. (11) that

$$\Gamma + \int_{-1/2}^{1/2} (|Dh_z|^2 + a^2\,|h_z|^2)\,dz < \frac{1}{\pi^2}\int_{-1/2}^{1/2} |Dw|^2\,dz. \tag{42}$$

Equation (39) upon using inequalities (41) and (42) gives

$$\frac{1}{\sigma}\left(1-\frac{R_2\sigma}{2\tau^2\pi^4}-\frac{Q\sigma_1}{\pi^2}\right)\int_{-1/2}^{1/2}|Dw|^2\,dz + \frac{a^2}{\sigma}\int_{-1/2}^{1/2}|w|^2\,dz + R_1 a^2\int_{-1/2}^{1/2}|\theta|^2\,dz < 0. \tag{43}$$

Inequality (43) obviously cannot hold under the conditions of the theorem. Hence $p_i = 0$.

This completes the proof of the theorem. It is to be noted that in deriving inequalities (40) and (42) the condition $p_r \geqslant 0$ has been used.

Theorem 3 implies that for the problem under consideration an arbitrary neutral ($p_r = 0$) or unstable ($p_r > 0$) mode of the system is definitely non-oscillatory in character and therefore in particular PES is valid if $R_2\sigma/2\tau^2\pi^4 + Q\sigma_1/\pi^2 \leqslant 1$.

(b) *An Exact Solution of the Problem*

THEOREM 4. *An exact solution of Eqs.* (8)–(11) *subject to the boundary conditions* (12) *is given by*

$$\begin{aligned} w = {} & \sum_{n=0}^{\infty}\frac{C_n(-1)^{n+1}}{16(2n+1)\pi}\left\{(2n+1)^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right\} \\ & \times\left[\{(2n+1)^2-5^2\}\sin 3\pi z+\{(2n+1)^2-3^2\}\sin 5\pi z\right] \\ & +\sum_{n=0}^{\infty}\frac{C_n}{(2n+1)\pi}\left\{(2n+1)^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right\}\sin(2n+1)\pi z, \end{aligned} \tag{44}$$

$$\begin{aligned} \theta = Bp\Bigg\langle & \sum_{n=0}^{\infty} C_n\frac{(-1)^{n+1}}{16(2n+1)\pi}\left\{(2n+1)^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right\} \\ & \times\left[\frac{\{(2n+1)^2-5^2\}}{\{\tau(3^2\pi^2+a^2)+p\}\{3^2\pi^2+a^2+p(1-\alpha_2T_0)\}}\sin 3\pi z\right. \\ & \left.+\frac{\{(2n+1)^2-3^2\}}{\{\tau(5^2\pi^2+a^2)+p\}\{5^2\pi^2+a^2+p(1-T_0\alpha_2)\}}\sin 5\pi z\right] \\ & +\sum_{n=0}^{\infty}\frac{C_n}{(2n+1)\pi} \\ & \times\frac{\{(2n+1)^2\pi^2+a^2+p\sigma_1/\sigma\}}{\{\tau(2n+1)^2\pi^2+\tau a^2+p\}\{(2n+1)^2\pi^2+a^2+p(1-T_0\alpha_2)\}} \\ & \times\sin(2n+1)\pi z\Bigg\rangle \end{aligned}$$

$$+\{(1-T_0\alpha_2)+B\}\left\langle\sum_{n=0}^{\infty}\frac{C_n(-1)^{n+1}}{16(2n+1)\pi}\left\{(2n+1)^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right\}\right.$$

$$\times\left[\frac{\{(2n+1)^2-5^2\}}{\{3^2\pi^2+a^2+p(1-T_0\alpha_2)\}}\sin 3\pi z\right.$$

$$\left.+\frac{\{(2n+1)^2-3^2\}}{\{5^2\pi^2+a^2+p(1-T_0\alpha_2)\}}\sin 5\pi z\right]$$

$$\left.+\sum_{n=0}^{\infty}\frac{C_n}{(2n+1)\pi}\frac{\{(2n+1)^2\pi^2+a^2+p\sigma_1/\sigma\}}{\{(2n+1)^2\pi^2+a^2+p(1-T_0\alpha_2)\}}\sin(2n+1)\pi z\right\rangle, \tag{45}$$

$$\phi=R_3\left\langle\sum_{n=0}^{\infty}\frac{C_n(-1)^{n+1}}{16(2n+1)\pi}\left\{(2n+1)^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right\}\right.$$

$$\times\left[\frac{\{(2n+1)^2-5^2\}}{\{\tau(3^2\pi^2+a^2)+p\}}\sin 3\pi z+\frac{\{(2n+1)^2-3^2\}}{\{\tau(5^2\pi^2+a^2)+p\}}\sin 5\pi z\right]$$

$$\left.+\sum_{n=0}^{\infty}\frac{C_n}{(2n+1)\pi}\frac{\{(2n+1)^2\pi^2+a^2+p\sigma_1/\sigma\}}{\{\tau(2n+1)^2+\tau a^2+p\}}\sin(2n+1)\pi z\right\rangle, \tag{46}$$

and

$$h_z=\sum_{n=0}^{\infty}\frac{3C_n(-1)^{n+1}}{16(2n+1)(3^2\pi^2+a^2+p\sigma_1/\sigma)}$$

$$\times\{(2n+1)^2-5^2\}\left\{(2n+1)^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right\}\cos 3\pi z$$

$$+\sum_{n=0}^{\infty}\frac{5C_n(-1)^{n+1}}{16(2n+1)(5^2\pi^2+a^2+p\sigma_1/\sigma)}$$

$$\times\{(2n+1)^2-3^2\}\left\{(2n+1)^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right\}\cos 5\pi z$$

$$+\sum_{n=0}^{\infty}C_n\cos(2n+1)\pi z, \tag{47}$$

together with the characteristic equation belonging to the lowest mode being given by

$$S_0=0, \tag{48}$$

where

$$B = T_0 \hat{\alpha}_2 R_3, \tag{49}$$

and

$$\begin{aligned} S_0 &= (\pi^2 + a^2)\{\pi^2 + a^2 + p(1 - T_0\alpha_2)\}\{\tau(\pi^2 + a^2) + p\} \\ &\times \left\{\left(\pi^2 + a^2 + \frac{p}{\sigma}\right)\left(\pi^2 + a^2 + \frac{p\sigma_1}{\sigma}\right) + Q\pi^2\right\} \\ &+ \left(\pi^2 + a^2 + \frac{p\sigma_1}{\sigma}\right)[R_2 a^2\{(\pi^2 + a^2 + p(1 - T_0\alpha_2))\} \\ &- R_1 a^2\{Bp + \langle \tau(\pi^2 + a^2) + p\rangle\langle (1 - T_0\alpha_2) + B\rangle\}]. \end{aligned} \tag{50}$$

Proof. Combining Eqs. (8)–(11) and boundary conditions (12) in an appropriate manner, we derive the following equation and boundary conditions in terms of w alone:

$$Lw = 0, \tag{51}$$

$$w = 0 = D^2 w = L_1 w = L_2 w = L_3 w \quad \text{at} \quad z = -\tfrac{1}{2} \text{ and } z = +\tfrac{1}{2}, \tag{52}$$

where

$$\begin{aligned} L &= (D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right)\left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right) \\ &\times \{\tau(D^2 - a^2) - p\}\{D^2 - a^2 - p(1 - T_0\alpha_2)\} \\ &- Q\, D^2(D^2 - a^2)\{\tau(D^2 - a^2) - p\}\{D^2 - a^2 - p(1 - T_0\alpha_2)\} \\ &- \left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right)\langle R_1 a^2[-Bp - \{\tau(D^2 - a^2) - p\}\{(1 - T_0\alpha_2) + B\} \\ &+ R_2 a^2\{D^2 - a^2 - p(1 - T_0\alpha_2)\}]\rangle, \end{aligned} \tag{53}$$

$$L_1 = D(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right) - QD^3 - \frac{Qp\sigma_1}{\sigma} D, \tag{54}$$

$$\begin{aligned} L_2 &= \tau\, D^3(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right) - Q\tau D^5 - \frac{Q\tau p\sigma_1}{\sigma} D^3 \\ &+ \left[R_1 a^2 \tau\{(1 - T_0\alpha_2) + B\} - R_2 a^2 - \frac{Q\tau p\sigma_1}{\sigma}\left(a^2 + \frac{p\sigma_1}{\sigma}\right)\right] D, \end{aligned} \tag{55}$$

and

$$L_3 = \tau^2 D^5(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right) - Q\tau^2 D^7 - Q\tau^2\left(a^2 + \frac{p\sigma_1}{\sigma}\right) D^5$$

$$- \left[R_1 a^2 \tau^2 \{(1 - T_0\alpha_2) + B\} + \tau R_2 a^2 + Q\tau^2 \left(a^2 + \frac{p\sigma_1}{\sigma}\right)^2 \right] D^3$$

$$+ \left[R_1 a^2 \tau^2 \{\langle a^2 + p(1 - T_0\alpha_2)\rangle \langle (1 - T_0\alpha_2) + B\rangle - Bp\tau\} \right.$$

$$\left. - R_2 a^2(\tau a^2 + p) - Q\tau^2\left(a^2 + \frac{p\sigma_1}{\sigma}\right)^3 \right] D. \tag{56}$$

The evenness of the operator L occurring in Eq. (51) and the identity of the boundary conditions that have to be satisfied at $z = \pm\frac{1}{2}$ as given by Eqs. (52) imply that the proper solution of Eq. (51) falls into two non-combining groups of even and odd solutions. Further, it follows from Eq. (10) that proper solutions for w and ϕ must either be both even or both odd while Eq. (11) implies that proper solutions for w and h_z must neither be both even nor both odd. From these considerations and the considerations of the corresponding hydrodynamic problem with dynamically free and thermally insulating boundaries it follows that the proper solutions for w, θ, and ϕ must be odd while that for h_z must be even. Therefore, if d_1 and d_2 are constants then the function

$$h_z - d_1 \cos 3\pi z - d_2 \cos 5\pi z$$

is even and since it is required to vanish at $z = \pm\frac{1}{2}$, we can expand it in a Fourier cosine series in the form

$$h_z - d_1 \cos 3\pi z - d_2 \cos 5\pi z = \sum_{n=0}^{\infty} C_n \cos(2n+1)\pi z. \tag{57}$$

With h_z given by Eq. (57), Eq. (11) becomes

$$Dw = d_1\left(3^2\pi^2 + a^2 + \frac{p\sigma_1}{\sigma}\right)\cos 3\pi z + d_2\left(5^2\pi^2 + a^2 + \frac{p\sigma_1}{\sigma}\right)\cos 5\pi z$$

$$+ \sum_{n=0}^{\infty} C_n\left\{(2n+1)^2\pi^2 + a^2 + \frac{p\sigma_1}{\sigma}\right\}\cos(2n+1)\pi z, \tag{58}$$

which upon integration yields

$$w=\frac{d_1}{3\pi}\left(3^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right)\sin 3\pi z+\frac{d_2}{5\pi}\left(5^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right)\sin 5\pi z$$

$$+d_3+\sum_{n=0}^{\infty}\frac{C_n}{(2n+1)\pi}\left\{(2n+1)^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right\}\sin(2n+1)\pi z, \tag{59}$$

where d_3 is a constant of integration. The requirement that the above solution for w satisfies the boundary conditions as specified by Eqs. (12) leads to a unique determination of d_1, d_2, and d_3 which are given by

$$d_1=\sum_{n=0}^{\infty}\frac{3(-1)^{n+1}C_n}{16(2n+1)(3^2\pi^2+a^2+p\sigma_1/\sigma)}$$

$$\times\{(2n+1)^2-5^2\}\left\{(2n+1)^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right\}, \tag{60}$$

$$d_2=\sum_{n=0}^{\infty}\frac{5(-1)^{n+1}C_n}{16(2n+1)(5^2\pi^2+a^2+p\sigma_1/\sigma)}$$

$$\times\{(2n+1)^2-3^2\}\left\{(2n+1)^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right\}, \tag{61}$$

and

$$d_3=0. \tag{62}$$

Substituting for d_1, d_2, and d_3 from Eqs. (60)–(62) in Eq. (59), we obtain a proper solution for w as

$$w=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}C_n}{16(2n+1)\pi}\left[(2n+1)^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right]$$

$$\times[\langle(2n+1)^2-5^2\rangle\sin 3\pi z+\langle(2n+1)^2-3^2\rangle\sin 5\pi z]$$

$$+\sum_{n=0}^{\infty}\frac{C_n}{(2n+1)\pi}\left\{(2n+1)^2\pi^2+a^2+\frac{p\sigma_1}{\sigma}\right\}\sin(2n+1)\pi z. \tag{63}$$

With w given by Eq. (63), Eq. (51) becomes

$$\sum_{n=0}^{\infty}C_n[\alpha_n S_1\sin 3\pi z+\beta_n S_2\sin 5\pi z]$$

$$+\sum_{n=0}^{\infty}C_n\gamma_n S_n\sin(2n+1)\pi z=0, \tag{64}$$

where

$$\alpha_n = \frac{(-1)^{n+1}}{16(2n+1)\pi}\left\{(2n+1)^2 - 5^2\right\}\left\{(2n+1)^2\pi^2 + a^2 + \frac{p\sigma_1}{\sigma}\right\}, \tag{65}$$

$$\beta_n = \frac{(-1)^{n+1}}{16(2n+1)\pi}\left\{(2n+1)^2 - 3^2\right\}\left\{(2n+1)^2\pi^2 + a^2 + \frac{p\sigma_1}{\sigma}\right\}, \tag{66}$$

$$\gamma_n = \frac{(2n+1)^2\pi^2 + a^2 + p\sigma_1/\sigma}{(2n+1)\pi}, \tag{67}$$

and

$$\begin{aligned} S_n = {} & \{(2n+1)^2\pi^2 + a^2\}\{(2n+1)^2\pi^2 + a^2 + p(1 - T_0\alpha_2)\} \\ & \times \{[(2n+1)^2\pi^2 + a^2]\tau + p\} \\ & \times \left\{\left[(2n+1)^2\pi^2 + a^2 + \frac{p}{\sigma}\right]\left[(2n+1)^2\pi^2 + a^2 + \frac{p\sigma_1}{\sigma}\right] + Q(2n+1)^2\pi^2\right\} \\ & + \left\{(2n+1)^2\pi^2 + a^2 + \frac{p\sigma_1}{\sigma}\right\}\langle R_2 a^2\{(2n+1)^2\pi^2 + a^2 + p(1 - T_0\alpha_2)\} \\ & - R_1 a^2\{Bp + [\tau\langle(2n+1)^2\pi^2 + a^2\rangle + p][(1 - T_0\alpha_2) + B]\}\rangle. \end{aligned} \tag{68}$$

Multiplying Eq. (64) by $\sin(2m+1)\pi z$ (since the first derivative with respect to z of the left hand side of Eq. (64) vanishes at $z = \pm\frac{1}{2}$) and integrating the resulting equation over the range of z, we get

$$\sum_{n=0}^{\infty} C_n[\alpha_n S_1 \delta_{1m} + \beta_n S_2 \delta_{2m} + \gamma_n S_n \delta_{nm}] = 0, \tag{69}$$

where $m = 0, 1, 2, 3, 4, ...,$ and δ_{nm} is Kronecker's delta. Equations (69) provide us with a set of linear homogeneous equations for the constants C_n's and the requirement that the determinant of this system of equations must vanish yields the characteristic equation for the determination of R_1 and p_i when $p_r = 0$. We thus obtain

$$\|\alpha_n S_1 \delta_{1m} + \beta_n S_2 \delta_{2m} + \gamma_n S_n \delta_{nm}\| = 0. \tag{70}$$

The nth approximation to the characteristic values of R_1 and p_i is obtained by setting the nth order determinant consisting of the first n rows and columns in the left hand side of Eq. (70) equal to zero, and this corresponds to the retention of the first n terms only in the Fourier expansion of $h_z - d_1 \cos 3\pi z - d_2 \cos 5\pi z$ as given by Eq. (57). We thus have

$$\begin{vmatrix} \gamma_0 S_0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \alpha_0 S_1 & (\alpha_1+\gamma_1)S_1 & 0 & \alpha_3 S_1 & \alpha_4 S_1 & \alpha_5 S_1 & \cdots & \alpha_{n-1} S_1 \\ \beta_0 S_2 & 0 & (\beta_2+\gamma_2)S_2 & \beta_3 S_2 & \beta_4 S_2 & \beta_5 S_2 & \cdots & \beta_{n-1} S_2 \\ 0 & 0 & 0 & \gamma_3 S_3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \gamma_4 S_4 & 0 & \cdots & 0 \\ & & & \cdots & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \gamma_{n-1} S_{n-1} \end{vmatrix} = 0. \tag{71}$$

Since α_n and β_n are non-zero numbers for every permissible value of n except $n=2$ and $n=1$, respectively, while γ_n does not vanish for any permissible value of n, it follows uniquely from Eq. (71) that the lowest characteristic value of R_1 and the associated value of $p_\imath$ are given by

$$S_0 = 0. \tag{72}$$

Further, since Eq. (72) is valid for all values of n, it follows that it is the unique solution that provides the lowest characteristic value of R and the associated value of $p_\imath$ as given by the characteristic equation (71).

With w as given by Eq. (63), θ and ϕ can be determined in accordance with Eqs. (9) and (10) together with the boundary conditions as specified by Eq. (12). Thus, we obtain the expressions for θ and ϕ as given by Eqs. (45) and (46), respectively. Further, substituting the values of d_1 and d_2 from Eqs. (60) and (61) in Eq. (57), we obtain the expression for h_z as given by Eq. (47). We now complete the solution of the problem by demonstrating that w, θ, ϕ, and h_z which are respectively given by Eqs. (44)–(47) and satisfy Eqs. (9)–(11) along with boundary conditions (12) also satisfy Eq. (8).

Equation (51) can be written in an alternative form as

$$\left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right)[D^2 - a^2 - p(1 - T_0\alpha_2)]E = 0, \tag{73}$$

where

$$p = \imath p_\imath, \; p_\imath \neq 0, \tag{74}$$

and

$$E = [\tau(D^2 - a^2) - p]\left[(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right)w - R_1 a^2\theta + \frac{R_2 a^2}{R_3}\phi + Q\,D(D^2 - a^2)h_z\right]. \tag{75}$$

For w, θ, ϕ, and h_z as given respectively by Eqs. (44)–(47), we have

$$DE = 0 = D^3 E \qquad \text{at} \quad z = -\tfrac{1}{2} \text{ and } z = +\tfrac{1}{2}. \tag{76}$$

Multiplying Eq. (73) by E^*, integrating over the range of z by parts appropriately with the help of Eqs. (76), and equating the imaginary part of the equation so obtained, we get

$$p_\iota\left[\frac{\sigma_1}{\sigma}+(1-T_0\alpha_2)\right]\int_{-1/2}^{1/2}(|DE|^2+a^2\,|E|^2)\,dz=0. \tag{77}$$

Since $p_\iota\neq 0$, it follows from Eq. (77) that

$$E=0, \qquad \forall z\in[-\tfrac{1}{2},\tfrac{1}{2}]. \tag{78}$$

Equation (78) can be written as

$$[\tau(D^2-a^2)-p]F=0, \qquad \forall z\in[-\tfrac{1}{2},\tfrac{1}{2}], \tag{79}$$

where

$$F=(D^2-a^2)\left(D^2-a^2-\frac{p}{\sigma}\right)w-R_1a^2\theta+\frac{R_2a^2}{R_3}\phi+Q\,D(D^2-a^2)h_z, \tag{80}$$

and

$$DF=0=D^3F \qquad \text{at} \quad z=-\tfrac{1}{2}\text{ and } z=+\tfrac{1}{2}. \tag{81}$$

Multiplying Eq. (79) by F^* and proceeding as before it follows that

$$p_\iota\int_{-1/2}^{1/2}|F|^2\,dz=0, \tag{82}$$

which in view of $p_\iota\neq 0$ implies that

$$F=0, \qquad \forall z\in[-\tfrac{1}{2},\tfrac{1}{2}]. \tag{83}$$

Equation (83) shows that Eq. (8) is satisfied.

This completes the proof of the theorem.

THEOREM 5. *For $p_r=0$ and $p_\iota\neq 0$, the characteristic equation belonging to the lowest mode, namely, Eq.* (48), *implies that*

$$\begin{aligned}R_1a^2=&\frac{1}{(\sigma+\tau\sigma_1)(1-T_0\alpha_2)+B\tau\sigma_1}\\&\times\Bigg[(\pi^2+a^2)\{Q\pi^2+(\pi^2+a^2)^2\}\{1+\tau(1-T_0\alpha_2)\}\sigma\\&+\tau(1+\sigma_1)(\pi^2+a^2)^3+R_2a^2\{\sigma_1+\sigma(1-T_0\alpha_2)\}\\&-\frac{p_\iota^2(\pi^2+a^2)}{\sigma}\{\sigma_1+\tau\sigma_1(1-T_0\alpha_2)+\sigma(1+\sigma_1)(1-T_0\alpha_2)\}\Bigg],\end{aligned} \tag{84}$$

and

$$A_1 p_i^4 + B_1 p_i^2 + C_1 = 0, \tag{85}$$

where

$$A_1 = \frac{(\pi^2+a^2)(1-T_0\alpha_2)\,\sigma_1^2\{-\tau B + 1 + \sigma(1-T_0\alpha_2)\}}{\sigma^2\{(\sigma+\tau\sigma_1)(1-T_0\alpha_2)+B\tau\sigma_1\}}, \tag{86}$$

$$\begin{aligned}
B_1 = {} & \frac{\tau\sigma_1}{\sigma^2}(\pi^2+a^2)^3 + \frac{(\pi^2+a^2)^3\,(1+\sigma_1)\{1+\tau(1-T_0\alpha_2)\}}{\sigma} \\
& + (1-T_0\alpha_2)(\pi^2+a^2)(Q\pi^2+\pi^2+a^2) + \frac{R_2 a^2\sigma_1(1-T_0\alpha_2)}{\sigma} \\
& + \frac{\tau(\pi^2+a^2)^3\,(\tau\sigma_1+\sigma+\sigma\sigma_1)\{-B-(1-T_0\alpha_2)\}}{\sigma\{(\sigma+\tau\sigma_1)(1-T_0\alpha_2)+B\tau\sigma_1\}} \\
& - \frac{\sigma_1(1-T_0\alpha_2)}{\sigma\{(\sigma+\tau\sigma_1)(1-T_0\alpha_2)+B\tau\sigma_1\}} \\
& \times [(\pi^2+a^2)\{Q\pi^2+(\pi^2+a^2)^2\}\{1+\tau(1-T_0\alpha_2)\}\sigma \\
& + \tau(1+\sigma_1)(\pi^2+a^2)^3 + R_2 a^2\{\sigma_1+\sigma(1-T_0\alpha_2)\}],
\end{aligned} \tag{87}$$

and

$$\begin{aligned}
C_1 = {} & -\tau(\pi^2+a^2)^3\,\{Q\pi^2+(\pi^2+a^2)^2\} - R_2 a^2(\pi^2+a^2)^2 \\
& - \frac{\tau(\pi^2+a^2)^2\,\{-B-(1-T_0\alpha_2)\}}{\{(\sigma+\tau\sigma_1)(1-T_0\alpha_2)+B\tau\sigma_1\}}\,[(\pi^2+a^2)\{Q\pi^2+(\pi^2+a^2)^2\} \\
& \times \{1+\tau(1-T_0\alpha_2)\}\sigma + \tau(1+\sigma_1)(\pi^2+a^2)^3 \\
& + R_2 a^2\{\sigma_1+\sigma(1-T_0\alpha_2)\}].
\end{aligned} \tag{88}$$

Proof. Putting $p = \imath p_i$, $p_i \neq 0$ in Eq. (48), separating the real and imaginary parts of the equation so obtained, and substituting for $R_1 a^2$ from the first of the resulting equations in the second equation, we get the result.

This completes the proof of the theorem.

Theorem 5 shows that overstable solutions do exist when both the bounding surfaces are dynamically free and provides us with the exact calculations for the critical Rayleigh number and the frequency of oscillations of the overstable motions at the marginal state with respect to them. It is important to note in this connection that in Banerjee's generalized Bénard model [11] under magnetic effects [7], it is only the overstable motions that manifest at the marginal state while in the simple

Bénard model under magnetic effects, or in Veronis' thermohaline model under magnetic effects, the possibility of both, the stationary as well as the overstable motions exist at the marginal state. However, Veronis' work gives ample support to the proposition that overstable motions at the marginal state are the most likely ones. It is on the basis of the above works of Banerjee and Veronis which are carried out for the case when both the bounding surfaces are dynamically free that we have taken $p_i \neq 0$ and $p_r = 0$ in Theorem 5 although solutions with $p_i = 0$ when $p_r = 0$ also exist as they exist in the simple Bénard model under magnetic effects.

THEOREM 6. *Under the hypothesis of Theorem* 5, *if* $\tau(1 - T_0\alpha_2) > 1$, $\sigma_1 \leqslant \sigma(1 - T_0\alpha_2)$, *and* $\hat{\alpha}_2 = 0$, *then*

$$p_i = 0.$$

Proof. Letting

$$\begin{aligned} x &= \frac{a^2}{\pi^2}, \qquad \imath p_i = \frac{p}{\pi^2}\ (p_i \text{ is real}), \qquad R_1^* = \frac{R_1}{\pi^4}, \\ R_2^* &= \frac{R_2}{\pi^4}, \qquad Q_1 = \frac{Q}{\pi^2}, \qquad \text{and} \qquad \hat{\alpha}_2 = 0, \end{aligned} \tag{89}$$

in Eq. (48), rearranging the various terms of the equation so obtained appropriately and separating the real and imaginary parts of the resulting equation, we get

$$\begin{aligned} &\frac{R_1^*(1 - T_0\alpha_2)}{[(1+x)^2 + p_i^2(1 - T_0\alpha_2)]} \\ &\quad = \frac{R_2^*\tau}{[\tau^2(1+x)^2 + p_i^2]} + \frac{1+x}{x}\left[1 + \frac{Q_1}{(1+x)^2 + p_i^2(\sigma_1^2/\sigma^2)}\right], \end{aligned} \tag{90}$$

and

$$\begin{aligned} &\frac{R_1^*(1 - T_0\alpha_2)^2}{(1+x)^2 + p_i^2(1 - T_0\alpha_2)^2} \\ &\quad = \frac{R_2^*}{\tau^2(1+x)^2 + p_i^2} + \frac{1+x}{\sigma x}\left\{\frac{Q_1\sigma_1}{(1+x)^2 + p_i^2(\sigma_1^2/\sigma^2)} - 1\right\}. \end{aligned} \tag{91}$$

Eliminating R_1^* between Eqs. (90) and (91), we get

$$\begin{aligned} &\frac{R_2^*\{\tau(1 - T_0\alpha_2) - 1\}}{\tau^2(1+x)^2 + p_i^2} \\ &\quad = \frac{Q_1(1+x)\{\sigma_1 - \sigma(1 - T_0\alpha_2)\}}{\sigma x\{(1+x)^2 + p_i^2(\sigma_1^2/\sigma^2)\}} - \frac{1+x}{\sigma x}\{1 + \sigma(1 - T_0\alpha_2)\}. \end{aligned} \tag{92}$$

Equation (92) obviously cannot hold under the conditions of the theorem for otherwise p_i would be purely imaginary which is contrary to the hypothesis that p_i is real. Hence, under the conditions of the theorem we must have

$$p_i = 0.$$

This completes the proof of the theorem.

Theorem 6 shows that PES is valid for the problem under consideration when both the bounding surfaces are dynamically free, electrically perfectly conducting, thermal, and concentration-wise non-conducting and $\tau(1 - T_0\alpha_2) > 1$, $\sigma_1 \leqslant \sigma(1 - T_0\alpha_2)$, and $\hat{\alpha}_2 = 0$. Further, it provides a natural extension of the sufficient conditions for the validity of PES

(i) in Veronis' thermohaline configuration (i.e., $\tau > 1$) and

(ii) Thompson–Chandrasekhar's criterion (i.e., $\sigma_1 \leqslant \sigma$) for the simple magnetohydrodynamic Bénard convection to the present modified framework.

THEOREM 7. *Under the hypothesis of Theorem* 5, *if* $\tau(1 - T_0\alpha_2) \leqslant 1$, $\sigma_1 > \sigma(1 - T_0\alpha_2)$, $\hat{\alpha}_2 = 0$, *and* $Q_1[\sigma_1 - \sigma(1 - T_0\alpha_2)] + R_2^*[1 - \tau(1 - T_0\alpha_2)]$ $\sigma/2\tau^2 \leqslant [1 + \sigma(1 - T_0\alpha_2)]$, *then*

$$p_i = 0.$$

Proof. Equation (92) can be written as

$$[1 + \sigma(1 - T_0\alpha_2)] = \frac{Q_1[\sigma_1 - \sigma(1 - T_0\alpha_2)]}{(1 + x)^2 + p_i^2(\sigma_1^2/\sigma^2)} + \frac{R_2^*\sigma x[1 - \tau(1 - T_0\alpha_2)]}{(1 + x)[\tau^2(1 + x)^2 + p_i^2]}. \quad (93)$$

Since $\sigma_1 - \sigma(1 - T_0\alpha_2) > 0$ and $1 - \tau(1 - T_0\alpha_2) \geqslant 0$, it follows from Eq. (93) that

$$[1 + \sigma(1 - T_0\alpha_2)] < Q_1[\sigma_1 - \sigma(1 - T_0\alpha_2)] + \frac{R_2^*\sigma[1 - \tau(1 - T_0\alpha_2)]}{2\tau^2}, \quad (94)$$

which is contrary to the given hypothesis of the theorem. Hence, under the conditions of the theorem, we must have

$$p_i = 0.$$

This completes the proof of the theorem.

The essential content of Theorem 7 is similar to that of Theorem 6. However, it provides us with an alternative sufficient condition for the validity of PES when the conditions of Theorem 6 are violated.

(c) *A Semi-circle Theorem for Arresting the Complex Growth Rate*

THEOREM 8. *If* $(p, w, \theta, \phi, h_z)$, $p = p_r + ip_i$, $p_r \geqslant 0$, $p_i \neq 0$, *is a solution of Eqs.* (8)–(11) *with* $\hat{\alpha}_2 = 0$ *and any of the boundary conditions* (12)–(19), *then*

$$|p|^2 < \max(R_2\sigma, Q^2\sigma^2). \tag{95}$$

Proof. For $\hat{\alpha}_2 = 0$, $p_i \neq 0$, and any of boundary conditions (12)–(19), Eq. (38) holds, i.e.,

$$\frac{1}{\sigma}\int_{-1/2}^{1/2} (|Dw|^2 + a^2 |w|^2)\, dz + R_1 a^2 \int_{-1/2}^{1/2} |\theta|^2\, dz$$
$$= \frac{R_2 a^2}{R_3^2}\int_{-1/2}^{1/2} |\phi|^2\, dz + \frac{Q\sigma_1}{\sigma}\left[\Gamma + \int_{-1/2}^{1/2} (|Dh_z|^2 + a^2 |h_z|^2)\, dz\right]. \tag{96}$$

Equations (10) and (11) when multiplied by their complex conjugates and integrated over the range of z yield upon utilizing $p_r \geqslant 0$ and any of the boundary conditions on ϕ and h_z as given in Eqs. (12)–(19) the following inequalities:

$$\int_{-1/2}^{1/2} |\phi|^2\, dz < \frac{R_3^2}{|p|^2}\int_{-1/2}^{1/2} |w|^2\, dz, \tag{97}$$

$$\int_{-1/2}^{1/2} |(D^2 - a^2)h_z|^2\, dz < \int_{-1/2}^{1/2} |Dw|^2\, dz, \tag{98}$$

and

$$\int_{-1/2}^{1/2} |h_z|^2\, dz < \frac{\sigma^2}{\sigma_1^2 |p|^2}\int_{-1/2}^{1/2} |Dw|^2\, dz. \tag{99}$$

Now,

$$\Gamma + \int_{-1/2}^{1/2} (|Dh_z|^2 + a^2 |h_z|^2)\, dz$$
$$= -\int_{-1/2}^{1/2} h_z^*(D^2 - a^2)h_z\, dz \leqslant \int_{-1/2}^{1/2} |h_z|\, |(D^2 - a^2)h_z|\, dz$$
$$\leqslant \left\{\int_{-1/2}^{1/2} |h_z|^2\, dz\right\}^{1/2} \left\{\int_{-1/2}^{1/2} |(D^2 - a^2)h_z|^2\, dz\right\}^{1/2}$$

(Schwartz's inequality)

which upon utilizing inequalities (98)–(99) gives

$$\Gamma + \int_{-1/2}^{1/2} (|Dh_z|^2 + a^2 |h_z|^2)\, dz < \frac{\sigma}{\sigma_1 |p|}\int_{-1/2}^{1/2} |Dw|^2\, dz. \tag{100}$$

Equation (96) together with inequalities (97) and (100) implies that

$$\left(\frac{1}{\sigma}-\frac{Q}{|p|}\right)\int_{-1/2}^{1/2}|Dw|^2\,dz+a^2\left(\frac{1}{\sigma}-\frac{R_2}{|p|^2}\right)$$
$$\times\int_{-1/2}^{1/2}|w|^2\,dz+R_1a^2\int_{-1/2}^{1/2}|\theta|^2\,dz<0. \tag{101}$$

It follows from inequality (101) that

$$|p|^2<\max(R_2\sigma,\,Q^2\sigma^2). \tag{102}$$

This completes the proof of the theorem.

Theorem 8 shows that the complex growth rate p $(=p_r+ip_i)$ of an arbitrary oscillatory $(p_i\neq 0)$ perturbation, neutral $(p_r=0)$ or unstable $(p_r>0)$, lies within a semi-circle with center at the origin and $(\text{radius})^2=\max(R_2\sigma,\,Q^2\sigma^2)$ in the right half of the $p_r\,p_i$-plane. Further, this result is valid for quite general boundary conditions.

Remarks. (1) Results for the various problems cited in Section 3 could be obtained from the present analysis by equating to zero the relevant parameters.

(2) Results analogous to those contained in Theorems 1–8 could also be derived for the modified hydromagnetic thermohaline convection of Stern's [12] type, namely the eigenvalue problem (8)–(19) with $R_1<0$ and $R_2<0$, by following the method of analysis adopted in the derivation of these theorems.

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